Bitcoin in Insurance Assets: Constructing a Risk-Neutral Economic Scenario Generator

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Abstract

Given the growing emergence of crypto-assets and their potential integration into life insurance savings products, this paper develops a risk-neutral Economic Scenario Generator (ESG) for bitcoin. Based on a Heston model calibrated using bitcoin options, our ESG cleverly combines Milstein and Quadratic Exponential (QE) diffusion schemes to ensure trajectory stability while satisfying market-consistency and martingality tests. Our methodology provides insurers with a robust tool for modeling bitcoin within the Solvency II framework, complementing our previous study that recommended an 84% shock on bitcoin for SCR calculation under standard formula approach.

Keywords: Life Insurance, Solvency II, Bitcoin, Crypto-assets, Calibration, Economic Scenario Generator.

Introduction

Within the Solvency II regulatory framework, effectively managing insurance contract portfolios demands sophisticated modeling of how risk factors will evolve over time. Insurers rely on Economic Scenario Generators (ESGs) that operate in a risk-neutral universe to create economic trajectories reflecting market expectations. Solvency II mandates that insurers value their liabilities using a "best-estimate" economic approach—calculating the liability's replication price by averaging future cash flows across various risk-neutral scenarios, adjusted for the time value of money. These risk-neutral ESGs are therefore critical tools that enable insurers to project the future cash flows necessary for accurately assessing the economic value of their obligations.

This paper presents the development of a specialized risk-neutral economic scenario generator (ESG) for bitcoin, created to produce insurance liability valuations that both align with market-observed prices and satisfy the regulatory martingale requirements. Our work builds upon previous research titled "What bitcoin shock should be applied for Solvency II SCR calculation?", which established a recommended 84% market shock parameter for bitcoin.

As emphasized by the French Prudential Supervision and Resolution Authority (ACPR), "The ESG must allow modeling of different types of assets held by the organization. The risk factors modeled are therefore based on its risk profile and must reflect the different sources of volatility to which the organization is exposed." ACPR 2020. Then, it is essential that the selected models correctly reflect the complexity of the underlying risks without underestimating technical provisions. Although simpler models are often preferred for their ease of use and understanding, it is imperative that their calibration correctly captures the implied volatility of assets, as validated by Market Consistency tests.

The development of a risk-neutral ESG requires calibrating asset valuation models against derivative product prices, including options and futures. These financial instruments embed market expectations and reveal the volatility patterns of underlying assets. Bitcoin presents a particular challenge due to its exceptional volatility, demanding a methodical approach that balances accurate volatility modeling with stable, dependable projections. This stability is especially critical because stochastic volatility models tend to produce potentially explosive trajectories, a tendency exacerbated by bitcoin's inherent price volatility. This paper addresses precisely this challenge by introducing a methodology specifically designed for this complex modeling problem.

Several models exist for evaluating derivative products. Among others, D. B. Madan, Reyners, and Schoutens 2019 presents an overview ranging from classic models like Black-Scholes to more sophisticated models such as VG-CIR (Variance Gamma - Cox–Ingersoll–Ross), capable of modeling jumps and stochastic volatility. Although VG-CIR offers a more realistic representation of markets, its complexity due to the calibration of eight parameters makes it a difficult option to use in a practical framework. Conversely, the Heston model, with only five parameters, constitutes an effective compromise between precision and simplicity of implementation.

What distinguishes our methodology is its novel integration of the Heston model with two complementary discretization schemes: Milstein and Quadratic Exponential (QE). This combination significantly enhances the stability of long-term projections, rendering our bitcoin-specific risk-neutral ESG particularly effective for modeling the cryptocurrency's complex and volatile behavior. Where traditional approaches rely on static or simplified models, our method successfully captures bitcoin's stochastic volatility characteristics while maintaining full compliance with Solvency II regulatory standards.

In this paper, we concentrate on developing a risk-neutral economic scenario generator founded on the Heston model for simulating bitcoin price trajectories. A subsequent article will examine the practical applications of this model to key Solvency II metrics, including Best Estimate Liability (BEL), Value In Force (VIF), and Solvency Capital Requirement (SCR).

This paper is organized as follows. We begin by introducing the Heston model and its calibration process using bitcoin option market data. We then elaborate on our discretization methodologies and trajectory simulation techniques, with particular focus on the Milstein and Quadratic Exponential schemes. The final section validates our approach through the regulatory testing framework mandated by Solvency II, demonstrating the effectiveness of our ESG for evaluating bitcoin-related risk exposures in insurance portfolios.

1 Heston Model and Calibration

Modeling bitcoin price dynamics necessitates a methodological approach capable of capturing its high volatility and distinctive stochastic properties. The Heston stochastic volatility framework provides an appropriate mathematical foundation for addressing this challenge. To implement this model, we calibrated its parameters using market-observed Bitcoin call option prices across various maturities and strike prices. We obtained these data from Deribit, a leading cryptocurrency derivatives exchange specializing in futures contracts and options on Bitcoin (BTC) and Ethereum (ETH). Established in 2016, this platform is characterized by substantial liquidity and primarily serves professional market participants.

1.1 Calibration Data

Figure 1.1 presents the Bitcoin call option data selected for our model calibration.

n°	strike	type	Maturity_day	Maturity_year	Spot	Price
1	26000	Call	4	0,0109589	28 479	2 612
2	26500	Call	4	0,0109589	28 479	2 162
3	27000	Call	4	0,0109589	28 479	1 746
4	27500	Call	4	0,0109589	28 479	1 361
5	28000	Call	4	0,0109589	28 479	1 022
6	28500	Call	4	0,0109589	28 479	752
7	29000	Call	4	0,0109589	28 479	544
.	.	.	.	.	.	.
.	.	.	.	.	.	.
63	31000	Call	270	0,73972603	28 479	4 978
64	26000	Call	361	0,9890411	28 479	7 946
65	27000	Call	361	0,9890411	28 479	7 504
66	28000	Call	361	0,9890411	28 479	7 088
67	29000	Call	361	0,9890411	28 479	6 713
68	30000	Call	361	0,9890411	28 479	6 351
69	31000	Call	361	0,9890411	28 479	5 981

Figure 1.1 **Bitcoin spot call option data quoted on April 14, 2023**

The selected options have maturities under one year, expressed in days, with a maximum duration of 361 days. This constraint reflects the highly volatile and speculative nature of cryptocurrency markets, where investment horizons and market expectations typically operate on shorter time-frames. These structural market characteristics justify the adoption of shorter maturities to accurately model the underlying dynamics. The final dataset [1] comprises 69 at-the-money call option observations [2] (see 1.1). At-the-money options exhibit enhanced sensitivity to price fluctuations in the underlying asset compared to out-of-the-money alternatives, thereby providing more informative signals regarding market-implied volatility—a critical parameter for calibrating stochastic volatility models within a risk-neutral framework.

Following calibration, we implemented scenario projections using a dual diffusion model approach, combining the Milstein scheme and the Quadratic Exponential method for volatility modeling.
Subsequently, we validated the simulated scenarios through regulatory compliance testing: first, using market consistency tests to verify alignment between generated scenarios and observed market prices for Bitcoin call options; and second, applying martingality tests to confirm that the projected price trajectories satisfy the martingale property.

1.2 Heston Model Specification

The Heston model is a bivariate stochastic process that simultaneously characterizes asset price evolution and its volatility dynamics. Unlike the Black-Scholes framework, which assumes constant volatility, the Heston model incorporates time-varying stochastic volatility. Within this model, asset prices follow a geometric Brownian motion, while volatility is governed by a mean-reverting CIR (Cox–Ingersoll–Ross) process.
Under the risk-neutral measure, the price and volatility dynamics are described by a system of stochastic differential equations (SDEs), fully specified in Appendix A.

For European option valuation within this model, we employ the Fast Fourier Transform (FFT) methodology developed by Carr and D. Madan 1999, which efficiently computes option prices by leveraging the model's characteristic function.
The mathematical details of this approach are elaborated in Appendix A.

Parameter calibration was performed by minimizing the root mean square error (RMSE) between market-observed call option prices and their model-calculated counterparts.

[1] We restricted our analysis to at-the-money options exclusively

[2] An option is considered "at-the-money" when its strike price approximately equals the current price of the underlying asset

The calibration results indicate an average error margin of €38 in predicting observed prices. This deviation represents a relatively small proportion of typical call option prices, suggesting reasonable model accuracy. Nevertheless, further refinements may be warranted to enhance predictive precision, particularly given the heightened volatility characteristic of Bitcoin options.

Figure 1.2 illustrates the correspondence between theoretical prices—calculated using the European call valuation formula with optimized Heston parameters—and actual market prices. This visualization provides a qualitative assessment of parameter estimation quality. The evident alignment between model-generated and market-observed prices demonstrates the model's capacity to effectively capture market dynamics.

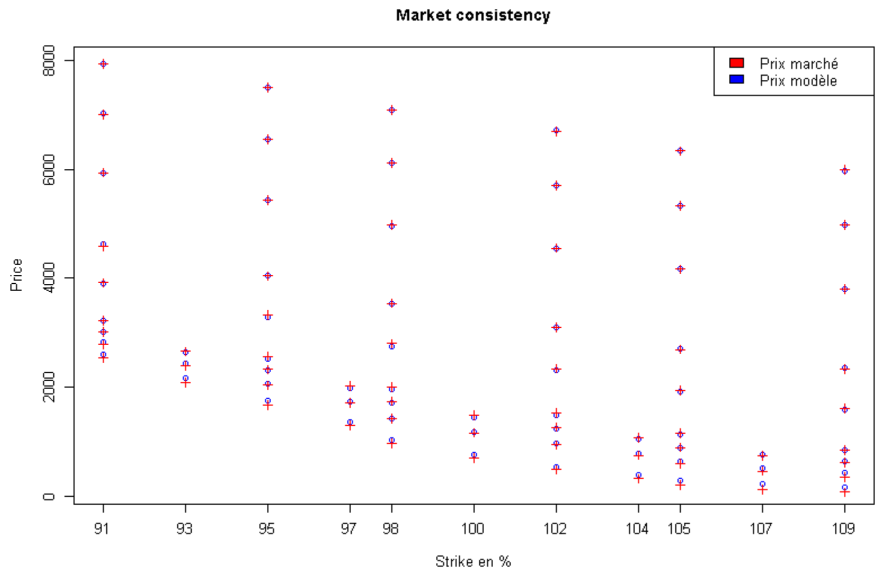

Figure 1.2 **Comparison of market-observed and theoretical prices**

2 Discretization and Trajectory Simulation

When generating price scenarios, selecting an appropriate simulation method for the evolution of the studied variable is essential. Various diffusion methods are proposed in the literature to reduce discretization error and address negative values in stochastic volatility models, which are both economically inconsistent and potential sources of numerical instability in simulations. Lord, Koekkoek, and Dijk 2010 suggests considering the positive part $\max(v_t, 0)$ of volatility v_t in the Heston model, concluding that this approach more effectively reduces bias compared to applying absolute value.

Milan and Jan n.d. compared several diffusion models, including Euler and Milstein schemes, and concluded that the Quadratic Exponential (QE) discretization scheme introduced by Andersen Andersen 2008 is the most accurate diffusion method for reproducing observed market prices of derivative products.
In our study, the parameters calibrated using bitcoin data, characterized by high volatility, necessitated adopting the Milstein scheme to prevent explosive trajectories in the short term.

However, for simulations over a 50-year horizon, the Milstein scheme proved insufficient, and the QE model was required to control trajectory explosion over this extended time frame.
Consequently, we combined these two diffusion models to produce a final model that simultaneously satisfies both market consistency and martingality tests.

2.1 Milstein and QE Schemes for Volatility Simulation

2.1.1 Milstein Diffusion Scheme

In our study, we implemented the Milstein scheme to simulate variance and price processes. This scheme is widely applied in finance for modeling various stochastic processes, particularly stochastic volatility models. The incorporation of an additional term in the Taylor expansion of the Euler scheme, as the Milstein approach does, enables superior approximation of geometric Brownian motion. This supplementary term enhances simulation precision by limiting extreme value occurrences in the volatility process—a crucial consideration for financial market modeling, where precise volatility management is essential for reliable forecasting.

It is important to acknowledge, however, that the Milstein scheme, like all numerical methods, is subject to approximation and discretization errors. Consequently, careful calibration is required, including sufficiently small discretization steps and a substantial number of scenarios to ensure trajectory stability. The mathematical implementation details and formulas are provided in Appendix B for comprehensive reference.

2.1.2 Quadratic Exponential Model (QE) for Volatility and Euler Scheme for Price Evolution

The Quadratic Exponential (QE) model enables volatility evolution estimation through approximation of the non-central χ^2 (chi-square) distribution. This method provides superior management of volatility fluctuations over time through a key parameter, ψ, which depends on the model's variance. In this framework, volatility at the subsequent time point depends on volatility at the previous time point, introducing significant dynamics into model evolution.

The determination of the next volatility value v_{n+1} employs two distinct approaches based on the value of ψ:

- When ψ is below a critical threshold ψ_C, future volatility is estimated by applying a transformation to a standard normal random variable.
- When ψ exceeds this critical threshold, an alternative approximation is employed, based on logarithmic transformation of a uniform random variable.

The threshold ψ_C serves as a pivotal element in the model, determining which approximation to implement. This parameter must lie between 1 and 2 to ensure proper model functionality. According to Milan and Jan n.d., a value of 1.5 is recommended.

The QE scheme addresses limitations in traditional discretization methods by handling the non-negative nature of variance processes more effectively. For values where $\psi \leq \psi_C$, the scheme applies a squared Gaussian approach modified to match the first two moments of the target distribution. When $\psi > \psi_C$, it employs an approximation based on an exponentially distributed random variable that preserves key distributional properties. This dual approach ensures numerical stability across different volatility regimes, making it particularly suitable for the extreme volatility conditions observed in cryptocurrency markets.

In our research, utilizing this combination of Milstein and QE schemes, we simulated bitcoin price trajectories and validated our models through regulatory tests mandated by Solvency II. The technical specifications and formulas employed in this method are detailed in Appendix C.

2.2 Selection of the Diffusion Model: Milstein Scheme and QE Discretization

The regulatory tests imposed by Solvency II directive—specifically the Market Consistency test and the martingality test—require discounting price trajectories using a risk-free rate curve. The Market Consistency test aims to ensure that derivative prices (aligned with observed financial market prices) are accurately reproduced by price trajectories simulated via Monte Carlo methods based on our Heston models and discretization schemes. The martingality test verifies that

the expectation of future discounted values of the underlying asset equals its initial value.
For our bitcoin ESG calibration, we employed the EIOPA zero-coupon risk-free rate curve published on October 31, 2023.
After conducting multiple tests, we observed two significant findings:

- The Milstein scheme demonstrated superior performance in reproducing market prices during the Market Consistency test. This test is critically important as it establishes the connection between an ESG and a specific asset. Consequently, maintaining the Milstein scheme was essential due to its capacity to faithfully reproduce market prices.
- The QE discretization produced near-martingale trajectories over a 50-year period.

However, neither scheme independently succeeded in generating trajectories that simultaneously satisfied both regulatory tests, despite our efforts to reduce the discretization step. Furthermore, we encountered computational limitations when attempting to decrease the discretization step while increasing the number of trajectories to enhance Monte Carlo precision. Our scenario martingalization approach initially involved diffusing prices using Milstein discretization until the point where the martingality test began to deteriorate. At this juncture, we extracted the Milstein prices and volatilities, which served as starting parameters for the QE discretization. Subsequently, we diffused the scenarios to the desired horizon using the QE scheme. This methodology is summarized in the diagram below.

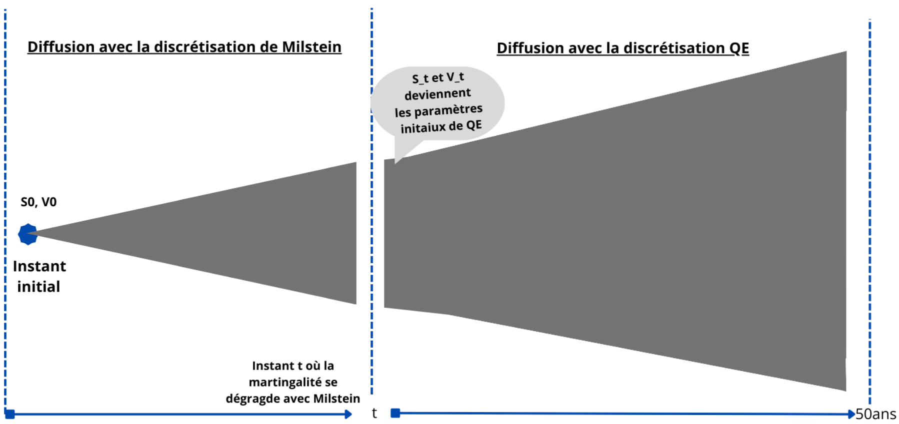

Figure 2.3 **Martingalization of price scenarios**

The method can be conceptualized as a two-stage martingalization process.

Stage 1: Determine the precise moment for merging the two schemes for 50-year diffusion.

We observed in Figure D.9 that the martingality test using the Milstein scheme begins to deteriorate after the tenth projection year. To precisely determine this temporal threshold where the martingality test degrades, we generated 1000 bitcoin price scenarios over a 15-year period, with a time step of 1/1000, employing Milstein discretization. Figure 2.4 illustrates the martingality test conducted on these bitcoin prices projected over 15 years.

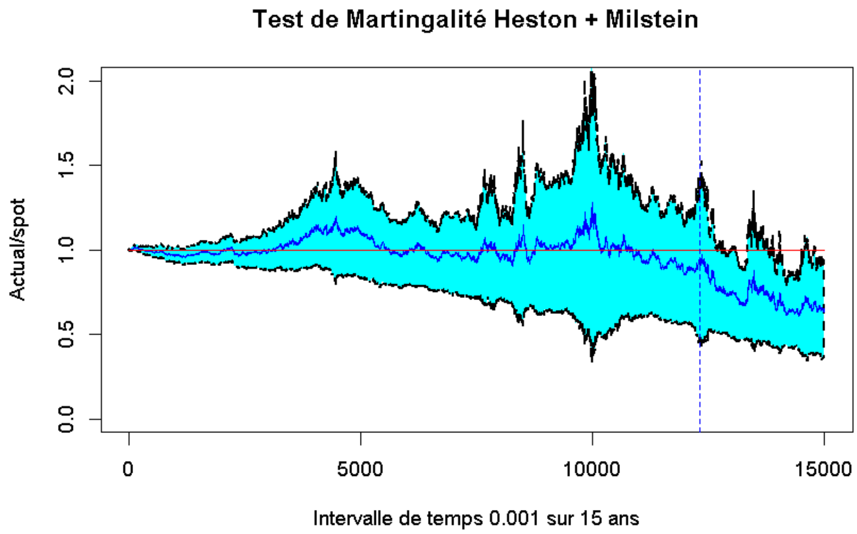

Figure 2.4 **Stage 1 of the martingalization process**

The vertical blue line in Figure 2.4 indicates the point we identified as the beginning of martingality test degradation, corresponding to 12.3 years. Consequently, the first 12 years of projection in our final ESG are executed using the Milstein discretization method.

Stage 2: Diffusion of scenarios over 50 years with the QE scheme

The prices $S_{t=12.3 \text{ years}}$ and volatilities $V_{t=12.3 \text{ years}}$ from the 1000 scenarios served as initial parameters for QE discretization. Subsequently, we projected bitcoin prices from year 12.3 to year 50. Following this projection, we merged the price trajectories from both discretization methods. Figure 2.5 demonstrates the martingality test applied to scenarios obtained after merging. To enhance visualization, the time step was increased to a weekly basis.

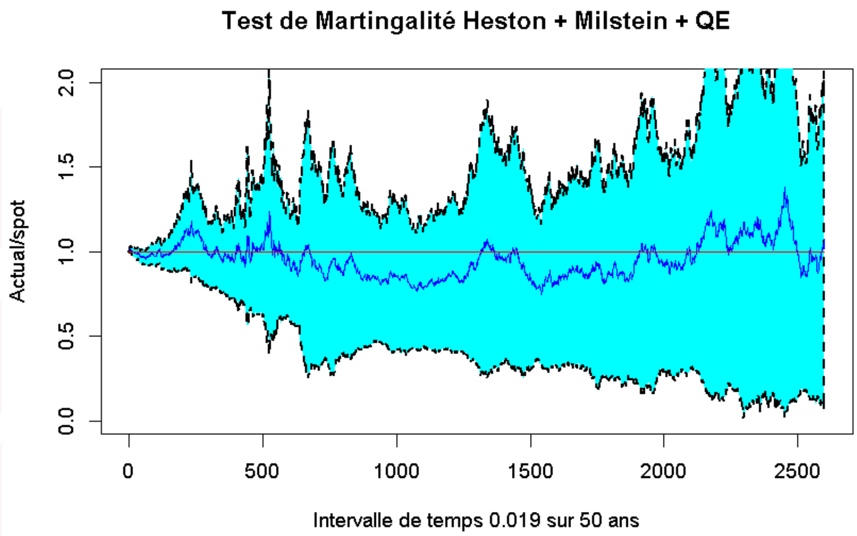

Figure 2.5 **Stage 2 of the martingalization process**

We observe that the martingality ratio (represented by the blue curve) remains remarkably close to 1 throughout the projection period and stays within the 95% confidence interval. Moreover, although the confidence interval tends to widen with increased projection duration, this widening is constrained, with the ratio primarily confined to the interval [0,2]. This outcome is deemed satisfactory with respect to our objective of controlling explosive trajectories.

3 Validations and Regulatory Tests

In the context of Asset-Liability Management (ALM) and the integration of Bitcoin into insurance portfolios, failure to validate regulatory tests presents several significant risks. First, inadequate scenario calibration can lead to erroneous risk projections, particularly by underestimating Bitcoin's extreme volatility, thereby affecting capital requirement estimations and provisions. This may also result in excessive market risk exposure, where the insurer would be underprepared for severe Bitcoin fluctuations, thus increasing the potential for substantial losses.

Lack of validation also exposes the insurer to regulatory non-compliance, risking sanctions, additional capital requirements, or operational restrictions, thereby compromising its solvency and competitive position. Finally, non-validation may underestimate liquidity impacts, especially during major market shocks, potentially impairing the insurer's ability to fulfill its obligations.

In summary, regulatory test validation is essential to ensure the reliability of financial projections, regulatory compliance, and financial resilience against risks associated with digital assets.

3.1 Scenario Validation with Regulatory Tests

In this final phase, we selected prices corresponding to annual time intervals in the price trajectory dynamics and re-performed the martingality test. Figure 3.6 illustrates the martingality test applied to scenarios generated using an annual time step.

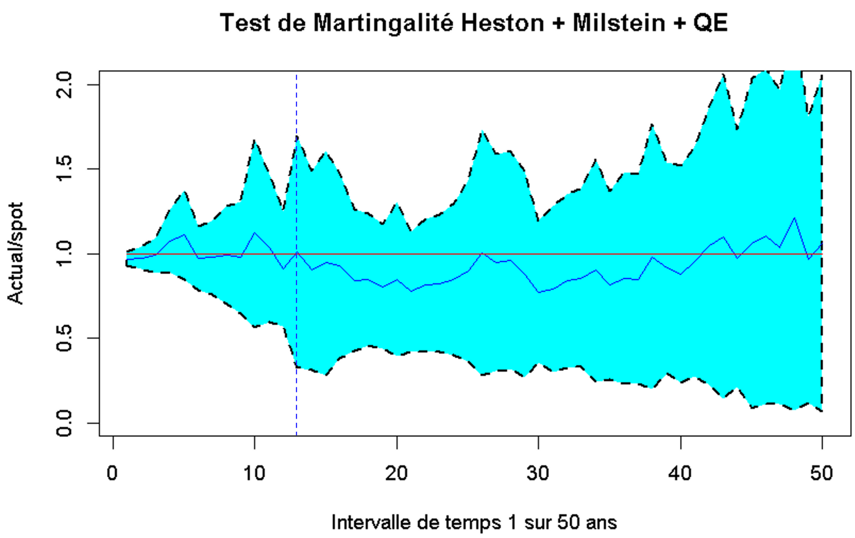

Figure 3.6 **Stage 3 of the martingalization process.**

With reference to the 95% confidence interval, we observe that the blue curve representing the martingality ratio consistently remains within the confidence bounds. This allows us to validate this test globally. The Market Consistency test is also verified by construction because the first 12 years of trajectories are derived from the Milstein scheme, which demonstrated satisfactory results for this test (see Table D.2). Therefore, our approach enables the generation of scenarios that simultaneously satisfy both Market Consistency and martingality tests over a 50-year horizon.

3.2 Regulatory Tests and Other Tested Models

Before selecting the final diffusion model (Milstein scheme with QE discretization), we conducted several tests to identify the optimal approach meeting the requirements of regulatory tests. The technical details and specific formulas used in this method are presented in Appendix D.

However, none of the studied schemes, taken in isolation, successfully generated trajectories that simultaneously satisfied both regulatory tests, despite reducing the discretization step. Additionally, we quickly encountered computational limitations. Indeed, to improve the precision of Monte Carlo simulations, it is necessary to reduce the discretization step and increase the number of trajectories, which imposes considerable constraints on computational power. These challenges highlight the complexity of the exercise and underscore the necessity for tailored solutions to fully meet regulatory requirements.

Furthermore, we considered the importance of market consistency and martingality tests, as well as the impact of discretization schemes on generated trajectories. In this context, we explored an innovative method, the "martingalization of price scenarios," which presents interesting potential for improving the generation of martingale trajectories.RéessayerClaude peut faire des erreurs. Assurez-vous de vérifier ses réponses.

Conclusion

The development of a risk-neutral Economic Scenario Generator (ESG) for bitcoin, presented in this study, represents a significant advancement in modeling the trajectories of this highly volatile asset within the Solvency II framework. By employing the Heston model, recognized for its ability to capture stochastic volatility, this ESG provides an innovative tool for assessing the risks associated with integrating bitcoin into insurance portfolios, generating trajectories that satisfy both market consistency and martingality tests.

The combination of Milstein and Quadratic Exponential (QE) diffusion schemes has enhanced the stability of long-term projections, thereby limiting the risk of explosive trajectories while maintaining modeling precision. Nevertheless, despite these advances, this study has revealed several significant limitations. The intrinsic volatility of bitcoin, coupled with short-term maturity derivative products (less than 1 year), makes parameter calibration complex, particularly for long-term investment horizons. Although robust within an initial framework, the projections could gain precision and reliability through more refined calibration methods and expanded computational capabilities.

Furthermore, while the QE scheme is relevant for limiting extreme values, it proves less effective at faithfully reproducing observed market prices, potentially introducing biases in projected prices for horizons beyond 13 years. Nevertheless, this study constitutes a valuable first step in evaluating the impact of crypto-assets in insurance, while highlighting the need for continued research to improve projection quality and perfect the integration of these assets into risk management strategies that comply with Solvency II requirements.

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A Heston Model

A.1 Model Structure

The Heston model is a two-factor model that describes the evolution of an underlying asset's price and its volatility. The asset price follows a geometric Brownian motion, while volatility follows a mean-reverting stochastic process, the CIR (Cox–Ingersoll–Ross) process. The model allows for time-varying volatility, in contrast to models that assume constant volatility, such as the Black-Scholes model.
Under the risk-neutral measure, the evolution of price and volatility is described by the following system of SDEs:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^S \tag{1}$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^V \tag{2}$$

where

- μ is the expected rate of return of the underlying asset
- κ is the mean reversion speed of volatility
- θ is the long-term volatility
- σ is the volatility of volatility
- W_t^S and W_t^V are Brownian motions with instantaneous correlation ρ, i.e., $Cov(dW_t^S, dW_t^V) = \rho dt$
- The initial conditions for price and volatility: $S_0 \geq 0$ and $V_0 \geq 0$

A.2 Option Valuation with the FFT Method

The valuation of European options in the Heston model can be performed using the Fast Fourier Transform (FFT). Commonly, when a model is too complex to obtain a closed-form pricing formula for derivative products, model prices are calculated using the Fast Fourier Transform (FFT) pricing method developed by Carr and D. Madan 1999.

Indeed, if the characteristic function $\phi(u)$ of the derivative product valuation model is analytically known, the price of a European Call option at date 0 with strike price K and time to maturity T is given by:

$$C(K,T) = e^{-rT} \int_0^\infty \mathrm{Re}\left(e^{-iu\ln(K)}\Phi(u-i)\right) du \tag{3}$$

where

- C is the Call price
- r is the risk-free interest rate
- K is the strike price
- T is the time to expiration
- i is the imaginary number

We also recall that, for identical maturity and strike price, it is possible to determine the price of a put option from the price of a call option using the Call-Put parity formula. This formula is expressed as follows:

$$\mathrm{Put} + \mathrm{S} = \mathrm{C} + \mathrm{K} \times e^{-rT}$$

The Fast Fourier Transform (FFT) method exploits the analytical form of the characteristic function to enable both rapid and efficient calculation.

The characteristic function $\phi_t(u)$ of $\log(S_t)$ for the Heston model is given by:

$$
\begin{aligned}
\phi_t(u) =& \exp\left(iu\log\left(S_0\right)\right)\\
&\times \exp\left(\theta\kappa\sigma^{-2}\left((\kappa-\rho\sigma ui-\eta)t-2\log\left((1-g\exp(-\mathrm{d}t))(1-g)^{-1}\right)\right)\right)\\
&\times \exp\left(V_0\sigma^{-2}(\kappa-\rho\sigma iu-\eta)(1-\exp(-\mathrm{d}t))(1-g\exp(-\mathrm{d}t))^{-1}\right)
\end{aligned}
$$

with

$$
\begin{aligned}
d &= \left((\rho\sigma ui-\kappa)^2-\sigma^2\left(-iu-u^2\right)\right)^{1/2},\\
g &= (\kappa-\rho\sigma ui-d)(\kappa-\rho\sigma ui+d)^{-1}.
\end{aligned}
$$

and $\kappa, \theta, \sigma, \rho, V_0$ the Heston parameters defined in section A.1.

A.3 Calibration of Heston Model Parameters

Calibrating the Heston model parameters is a crucial step to ensure that the calibrated parameters make the Heston model as close as possible to market observations. This step is fundamental to guarantee that simulations and valuations based on the Heston model are relevant and representative of economic reality. In this study, parameter adjustment is performed by minimizing the root mean square error (RMSE) between observed Call option prices in the market and those calculated by the model. The root mean square error function to minimize is given by:

$$
\mathrm{RMSE} = \sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}(P_{\mathrm{model},i}-P_{\mathrm{observed},i})^2\right)}
$$

where

- N is the total number of observed options,
- $P_{\mathrm{model},i}$ is the price of option i according to the Heston model,
- $P_{\mathrm{observed},i}$ is the observed price of option i in the market.

The objective is therefore to determine the optimal parameters $\hat{\theta}, \hat{\kappa}, \hat{\sigma}, \hat{\rho}, \hat{v}_0$ such that:

$$
\left(\hat{\theta}, \hat{\kappa}, \hat{\sigma}, \hat{\rho}, \hat{v}_0\right) = \arg\min_{(\theta,\kappa,\sigma,\rho,v_0)} \frac{1}{N}\sum_{i=1}^{i=N}\left(P_{\mathrm{model},i}-P_{\mathrm{observed},i}\right)^2, \tag{4}
$$

with Call_i the theoretical price.

During the optimization of these parameters, it is common to ensure that the adjusted parameters satisfy the Feller condition to guarantee the non-negativity of the volatility process. The Feller condition states that if the parameters satisfy the following condition, then the process V_t remains strictly positive almost surely:

$$
2\hat{\kappa}\hat{\theta} > \hat{\sigma}^2
$$

The optimal parameters, obtained following model adjustment and verifying the Feller condition, are presented in Table A.1:

algorithm	RMSE	κ	θ	σ	ρ	V_0
Nelder-Mead	38.507	1.302	0.546	1.192	-0.097	0.355

Table A.1 Optimal parameters of the Heston model

It is important to note that the observed mean reversion speed is relatively low, close to 1.3. Additionally, the volatility, represented by sigma, is very high, reaching 1.192. These factors highlight the high volatility of bitcoin and the uncertainty of its behavior, which is evident in the current market.

Moreover, a modest negative correlation (around -10%) was observed between the bitcoin price process and its volatility process.

However, despite the consistency of the estimated parameters, the model still presents limitations in its ability to accurately predict observed market prices. Indeed, the RMSE analysis reveals that the model has an average error of €38 in predicting observed prices. This error margin underscores the persistent challenges in modeling bitcoin and the complexity of its price movements.

Ultimately, although the error margin of €38 may seem significant in absolute terms, it is crucial to put it into perspective relative to the observed prices of bitcoin Call options. This error margin represents only a negligible fraction of the observed Call option prices. Thus, although the model may present limitations in its ability to accurately predict observed prices, this RMSE value remains acceptable in the context of evaluating bitcoin Call options with the Heston model in our study.

Figure A.7 illustrates the correspondence between theoretical prices—calculated via the evaluation formula (equation 3) of a European Call—and actual observed market prices. It provides a visual assessment of the precision with which the optimal parameters are adjusted. A notable adequacy of the model in reflecting market prices can be observed on the graph.

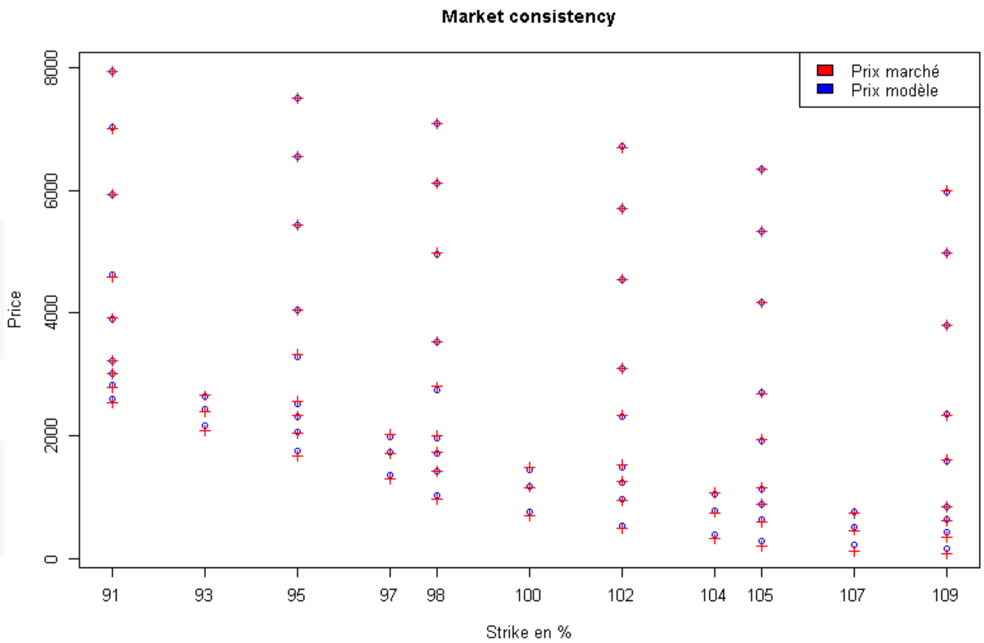

Figure A.7 Match between market prices and theoretical prices

B Milstein Diffusion Scheme

The Milstein scheme is an enhancement of the Euler scheme, incorporating an additional term through Itô's lemma and second-order Taylor series expansion. As indicated by Gatheral 2006, its application to the variance process reduces the frequency of negative values and discretization error compared to the Euler scheme. However, negative values may still appear, necessitating adjustments similar to those suggested by Lord, Koekkoek, and Dijk 2010, namely $v_n^+ = \max(v_n, 0)$.

Here is the Milstein scheme as adopted in our study:

$$v_{n+1} = \underbrace{v_n + \kappa\left(\theta - v_n^+\right)\Delta_n + \sigma\sqrt{v_n^+}Z_v\sqrt{\Delta_n}}_{\text{Euler Discretization}} + \frac{1}{4}\sigma^2\left(Z_v^2 - 1\right)\Delta_n,$$

where Z_v represents a standard normal random variable. Similarly, we can apply the Milstein scheme to the process S_n, which gives:

$$S_{n+1} = \underbrace{S_n + rS_n\Delta_n + S_n\sqrt{v_n^+}Z_S\sqrt{\Delta_n}}_{\text{Euler Discretization}} + \frac{1}{2}S_n v_n^+\left(Z_S^2 - 1\right)\Delta_n,$$

where Z_S is a standard normal random variable with correlation ρ to Z_v.

The Milstein scheme is widely used in finance for simulating various stochastic processes, including but not limited to stochastic volatility models. The addition of the extra term in the Milstein scheme, compared to the Euler scheme, allows for a better approximation of the geometric Brownian motion, which is particularly useful in the context of financial market modeling. Moreover, although the Milstein scheme can still produce negative values for the volatility process, the frequency of these occurrences is reduced, which improves the stability and accuracy of simulations. Finally, it is worth mentioning that the Milstein scheme, like any other numerical scheme, is subject to approximation and discretization errors and must therefore be calibrated with caution.

C Quadratic Exponential Model (QE) for Volatility

This model exploits the fact that the value v_{n+1}, conditioned by v_n, follows a non-central χ^2 (chi-square) distribution. It employs two distinct approximations of this distribution, depending on the variance values. This method is particularly useful for volatility management. The non-centrality parameter for v_{n+1} is proportional to v_n, and for high values of v_n, the model uses a power function on a standard normal random variable Z_v to estimate v_{n+1}. Consequently, a critical threshold $\psi_C \in [1, 2]$ is defined and compared to ψ, calculated as $\psi = s^2/m^2$, where m^2 and s^2 are functions of v_n:

$$m = \theta + (v_n - \theta)e^{-\kappa\Delta t} \tag{5}$$

and,

$$s^2 = \frac{v_n\sigma^2 e^{-\kappa\Delta t}}{\kappa}\left(1 - e^{-\kappa\Delta t}\right) + \frac{\sigma^2\theta}{2\kappa}\left(1 - e^{-\kappa\Delta t}\right)^2 \tag{6}$$

If U_v is defined as a uniform random variable and $Z_v = \phi^{-1}(U_v)$, where ϕ represents the cumulative distribution function of a normal distribution, then the two distinct approximations of the non-central χ^2 distribution, based on the values of volatility v_n, are specified as follows:

1. If $\psi \leq \psi_C$, we set $v_{n+1} = a\left(b + Z_V\right)^2$, where

$$a = \frac{m}{1+b^2}, \quad b^2 = 2\psi^{-1} - 1 + \sqrt{2\psi^{-1}}\sqrt{2\psi^{-1} - 1} \geq 0 \quad \text{and} \tag{7}$$

2. If $\psi > \psi_C$ we set $V_{n+1} = \Psi^{-1}(U_V; p, \beta)$, where

$$\Psi^{-1}(u) = \Psi^{-1}(u; p, \beta) = \begin{cases} 0 & \text{for } 0 \leq u \leq p \\ \beta^{-1} \ln\left(\frac{1-p}{1-u}\right) & \text{for } p < u \leq 1 \end{cases}, \tag{8}$$

where,

$$p = \frac{\psi - 1}{\psi + 1} \quad \text{and} \quad \beta = \frac{1-p}{m}.$$

As mentioned previously, the value of ψ_C determines which approximation to use. The first approximation is only valid for $\psi \leq 2$. If ψ exceeds this value, which corresponds to low values of v_n, the scheme will fail. When $\psi \geq 1$, the second approximation can be applied. Therefore, ψ_C must lie in the interval $[1, 2]$ to ensure the functioning of the discretization scheme. In the article by Milan and Jan n.d., the authors recommend a value of 1.5 for the threshold ψ_C.

In the context of our research, we combined the discretization methods of Milstein and QE. We chose the suggested value of 1.5 as the threshold ψ_C for the QE method, in accordance with the recommendation for the critical threshold ϕ_C. Using the optimal parameters of the Heston model, we simulated bitcoin price trajectories to generate our economic scenarios. The Solvency II standard imposes regulatory tests for the validation of stochastic models. These tests assist in decision-making for the selection of models for bitcoin price projection. In the following section, we will present the scenarios generated by these two schemes to the regulatory tests.

D Regulatory Tests

D.1 Market Consistency Test

In accordance with the S2 standard, it is essential to conduct a consistency test of the ESG derived from the model with the observed market data. The market consistency test, performed using the Monte Carlo method, serves to confirm that the prices produced by the model approximate the market prices of derivative options linked to the underlying asset. This market consistency test differs from that presented in Figure A.7. Indeed, the market consistency test via Monte Carlo is based on Call prices derived from model scenarios rather than on the closed-form evaluation formula of the Heston Call price. This test is essential to determine if the chosen diffusion model can create scenarios that represent with acceptable precision the reality of the market and its future forecasts.

To conduct this test, we generated 10,000 price trajectories with a time interval of $\frac{1}{1000}$ over a period of one year, using the Milstein and QE methods. Using the Monte Carlo method, we evaluated and compared the ability of the tested models to create price trajectories for bitcoin that reflect the market prices of the derivative products used. The calculation of European call option prices by the Monte Carlo method is formulated as follows:

$$C(K,T) = e^{-rT} \frac{1}{N} \sum_{i=1}^{N} (S_T^i - K)^+ \tag{9}$$

where,

- C is the price of the European call option,
- r is the risk-free interest rate,
- N is the number of Monte Carlo simulations,
- S_T^i is the price of the underlying asset at the expiration of the option in the i-th simulation,
- K is the strike price of the option, and

- $(x)^+$ is the function max(0, x), which represents the payoff of the option.

This formula calculates the option price by averaging the payoffs of the options for a large number of Monte Carlo simulations, discounted to the current time.
Given the popularity of the Euler method, we also included its RMSE to compare its performance with those of the Milstein and QE methods.
The RMSEs from the scenarios generated for each method are presented in Table D.2.

Discretization Method	RMSE
Euler	6221.5
QE	2779.9
Milstein	56.8

Table D.2 Comparison of diffusion models

According to the RMSE values (Table D.2), the Milstein method proved to be the most effective in generating prices close to those of the market among the evaluated methods. The Euler and QE methods show considerable RMSEs. However, the Euler scheme is the least performant with an RMSE of 6221.5.
Figure D.8 illustrates the Market Consistency test with the Milstein scheme.

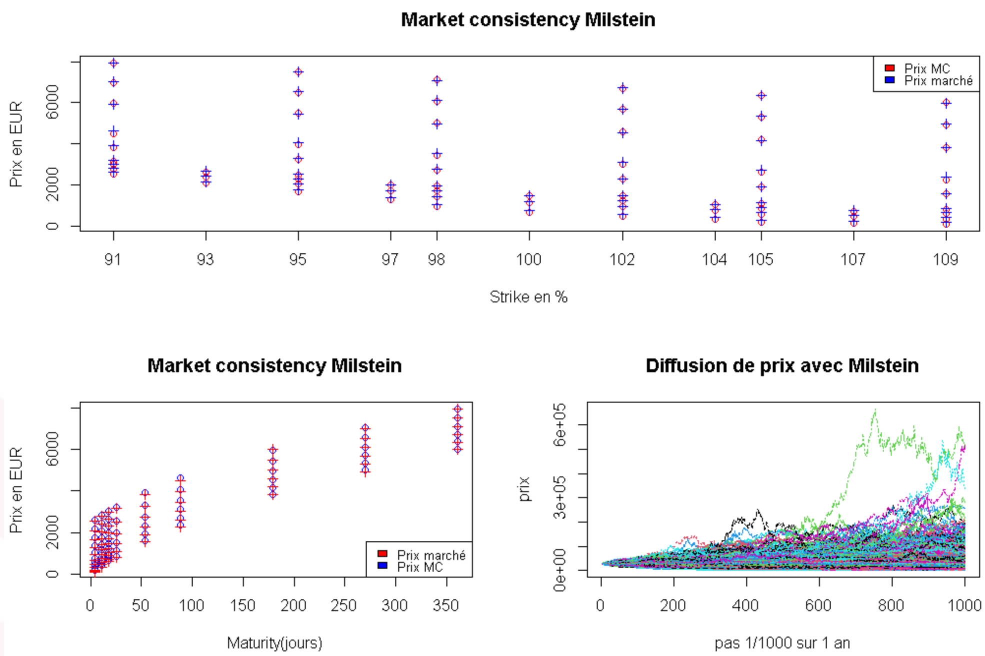

Figure D.8 Market Consistency test with Milstein

Figure D.8 illustrates the effectiveness of the Heston model, in combination with the Milstein diffusion scheme, in generating over a short horizon (1 year), bitcoin price trajectories that reproduce via Monte Carlo the prices of call options in our database.

D.2 Martingality Test

In the context of Pillar I of S2, the use of martingale trajectories in ALM modeling offers several specific advantages. In summary, the use of martingale trajectories calibrated in a risk-neutral world in ALM modeling offers a coherent and realistic approach to evaluate and manage the risks associated with the assets and liabilities of a financial institution. These models provide projections of future cash flows aligned with market expectations, which allows for effective risk management, portfolio optimization, and accurate evaluation of hedging strategies.

Theoretically, the test compares the initial price with the expectation of the sums of discounted prices at each period. If the process is a martingale, the ratio between the expectation of the sums of discounted prices and the initial price should be close to 1.

$$R = \frac{\mathbb{E}\left\{\exp\left[-\int_t^T r_u du\right] * P(t,T) \mid \mathcal{F}_t\right\}}{P(t,t)} \tag{10}$$

with:
- r_t the instantaneous short rate at time t
- $P(t,T)$ the price of the stock at time T seen at time t.

By applying the Heston model, we produced 1000 price scenarios for each discretization method (Milstein and QE) over a period of 50 years with a time step of 1/1000. The martingality ratios were calculated using formula 10. The results of the martingality tests for each method are shown below, with and without confidence intervals for clarity. The confidence level of the intervals is set at 95%.

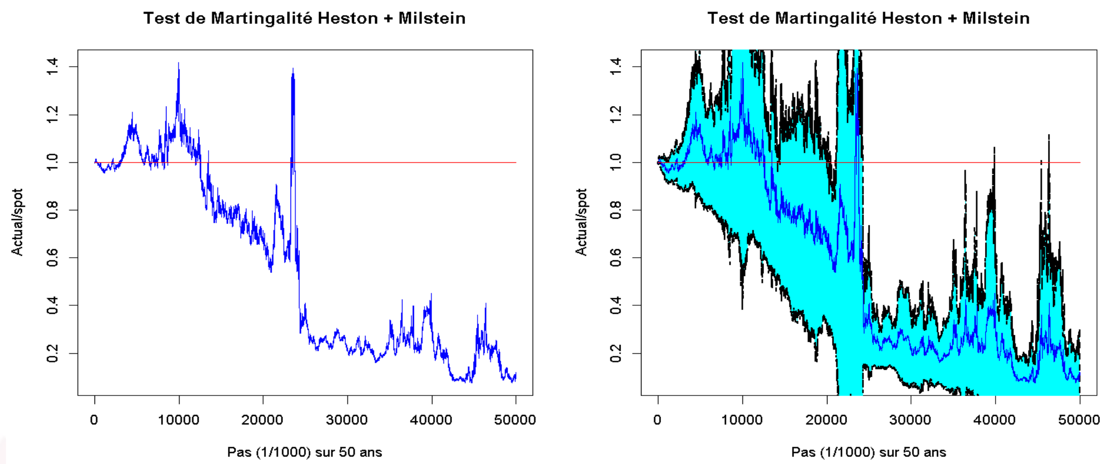

Figure D.9 Martingality test with Milstein

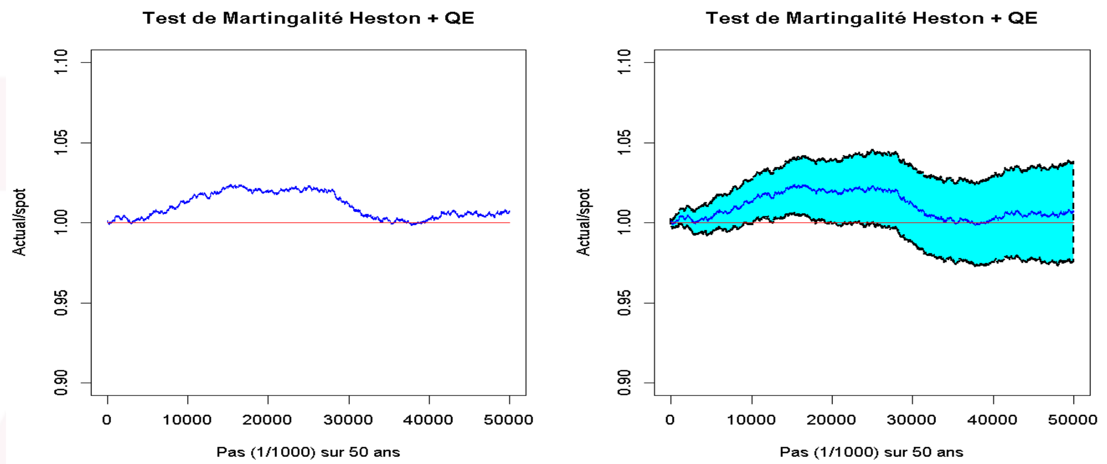

Figure D.10 Martingality test with QE

From the two Figures D.9 and D.10, we observe that if the Milstein discretization was previously the one that best reproduced the observed call option prices, the opposite occurs with the martingality test. Indeed, the high volatility of prices combined with discretization errors leads to a progressive cancellation of price trajectories during the projection, hence a ratio that becomes zero from the 10th year as illustrated in Figure D.9.

In contrast, the QE discretization, which models volatility using two approximations of the χ^2 distribution, allows for controlling the volatility process, thus avoiding extremes such as explosions or collapses of trajectories. However, as we noted with the Market Consistency test, this method has the disadvantage of masking certain aspects of market reality given that the generated scenarios do not faithfully reflect the observed market prices.

Nevertheless, the Solvency II (S2) standard requires that the ESGs input to the ALM model pass both market consistency and martingality tests when projecting assets in a risk-neutral universe. An innovative approach presented in this paper was to design a method that leverages the characteristics of both discretization schemes. We called this method the "martingalization of price scenarios." Although this method presented limitations, we used it to generate the bitcoin ESG. We are confident in its potential to perfect the creation of martingale trajectories for other financial assets.

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